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LETTERS

Simple Frequency Estimation via Exponential Samples

Steven Kay, *Fellow, IEEE*

Abstract—**A method for determining the frequency of a real sinusoid is proposed. Based on exponential sampling of the waveform, the approach requires virtually no computation. It can be easily implemented in digital hardware.**

I. Description of Method

Assume we observe the real continuous-time sinusoid $s(t) = A\sin(2\pi F_0 t)$, where $A > 0$ and that we wish to determine the frequency F_0, which satisfies $0 \le F_0 < F_{\mathrm{MAX}}$. The sinusoidal phase is assumed to be zero. Let $F_{\mathrm{MAX}} = 2^M$ for some integer M. To measure the frequency to an accuracy of N, we use

$$\hat{F}_0 = F_{\mathrm{MAX}} \sum_{n=1}^{N} b_n 2^{-n} \tag{1}$$

where

$$\begin{aligned} b_n &= 0 \quad \text{if} \quad s(t_n) > 0 \\ &= 1 \quad \text{if} \quad s(t_n) < 0 \end{aligned}$$

and $t_n = 2^{n-M-1}$. If $s(t_n) = 0$ for some $n = n_0$, then

$$\hat{F}_0 = F_{\mathrm{MAX}} \left(\sum_{n=1}^{n_0-1} b_n 2^{-n} + 2^{-n_0} \right) \tag{2}$$

which effectively terminates the expansion.

II. Rationale

Define the normalized frequency as $f_0 = (F_0/F_{\mathrm{MAX}})$ so that $0 \le f_0 < 1$. We use a binary expansion for f_0 as

$$f_0 = \sum_{n=1}^{\infty} b_n 2^{-n} \tag{3}$$

where $b_n = 0$ or 1.

For uniqueness, it is assumed that the dyadic rationals, that is, numbers of the form $k/2^L$ for L an integer and $k = 0, 1, \cdots, 2^L - 1$, are represented by *terminating* expansions. Hence, the number 3/4 is represented as $0.11000\cdots$ as opposed to $0.10111\cdots$. With (3), determination of f_0 is equivalent to determination of $\{b_n\}$. Now, consider the angle of the sinusoid or $\theta(t) = 2\pi F_0 t$. For sample times $t_n = 2^{n-M-1}$, the angle samples become

$$\begin{aligned} \theta_n &= 2\pi F_0 t_n \\ &= 2\pi f_0 F_{\mathrm{MAX}} 2^{n-M-1} \\ &= 2\pi f_0 2^{n-1} \end{aligned} \tag{4}$$

for $n = 1, 2, \cdots$. Consider $n = 1$ so that $\theta_1 = 2\pi f_0$. If $b_1 = 0$, then from (3), $0 \le f_0 < 1/2$. The upper limit of 1/2 is assumed not to be possible due to assumption of a terminating representation since 1/2 is represented as $0.1000\cdots$. Hence, $b_1 = 0$ if and only if $0 \le \theta_1 < \pi$. In addition, if $b_1 = 1, 1/2 \le f_0 < 1$ so that $b_1 = 1$ if and only if $\pi \le \theta_1 < 2\pi$. Next, consider $n = 2$ so that $\theta_2 = 4\pi f_0$. Using (3)

$$\begin{aligned} \theta_2 &= 4\pi \left(\frac{b_1}{2} + \sum_{n=2}^{\infty} b_n 2^{-n} \right) \\ &= 2\pi b_1 + 2\pi \left(\frac{b_2}{2} + \sum_{n=3}^{\infty} b_n 2^{-n+1} \right). \end{aligned}$$

We can omit the $2\pi b_1$ term since it equates to 0 or 2π, and thus, it does not affect the sine function. Let $\alpha = b_2/2 + \sum_{n=3}^{\infty} b_n 2^{-n+1}$ and note that if $b_2 = 0, 0 \le \alpha < 1/2$ and if $b_2 = 1, 1/2 \le \alpha < 1$. Hence

$$\begin{aligned} 0 \le \theta_2 < \pi \quad &\text{if and only if} \quad b_2 = 0 \\ \pi \le \theta_2 < 2\pi \quad &\text{if and only if} \quad b_2 = 1. \end{aligned}$$

By continuing the same argument, we can show that

$$\begin{aligned} 0 \le \theta_n < \pi &\Leftrightarrow b_n = 0 \\ \pi \le \theta_n < 2\pi &\Leftrightarrow b_n = 1. \end{aligned} \tag{5}$$

Excepting the case when $\theta_n = 0$ or π, this translates into

$$\begin{aligned} s(t_n) > 0 &\Leftrightarrow b_n = 0 \\ s(t_n) < 0 &\Leftrightarrow b_n = 1. \end{aligned}$$

If $s(t_{n_0}) = 0$ for some $n = n_0$, then $\theta_{n_0} = \pi k$ for k an integer. Thus, from (4)

$$\pi k = 2\pi f_0 2^{n_0 - 1}$$

or $f_0 = k/2^{n_0}$, which is a dyadic rational. It then follows that

$$\begin{aligned} \theta_n &= 2\pi f_0 2^{n-1} \\ &= 2\pi \frac{k}{2^{n_0}} 2^{n-1} \\ &= \pi k 2^{n-n_0} \end{aligned} \tag{6}$$

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which is a multiple of 2π for $n > n_0$, and hence, $b_n = 0$ for $n > n_0$ based on (5). If n_0 is the sample for which $s(t_n)$ is first equal to zero, then k must be odd. Otherwise, $k = 2l$, and from (6)

$$\theta_n = \pi l 2^{n-n_0+1}$$

so that $s(t_{n_0-1}) = 0$, contradicting the assumption of the first zero. In summary, if n_0 is the first sample term for which $s(t_n) = 0$, then we should choose $b_{n_0} = 1$ since $\theta_{n_0} = k\pi$ for k odd, and $b_n = 0$ for $n > n_0$ since θ_n will be a multiple of 2π.

As an example, assume $F_{\text{MAX}} = 64$ Hz and $F_0 = 60$ Hz so that $M = 6$.

For $N = 4$ bit accuracy, we sample at

$$\begin{aligned} t_n &= 2^{n-M-1} \qquad n = 1,2,3,4 \\ &= \tfrac{1}{64}, \tfrac{1}{32}, \tfrac{1}{16}, \tfrac{1}{8} \qquad \text{s.} \end{aligned}$$

Then, $s(t_n) = A\sin(2\pi(60)t_n)$ so that $s(t_1) < 0, s(t_2) < 0, s(t_3) < 0, s(t_4) = 0$. Applying (2) with $n_0 = 4$, we have

$$\hat{F}_0 = 64(2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}) = 60\,\text{Hz}.$$

III. Practical Considerations

A. Noise/Phase Errors

Assume that the sinusoidal phase is not zero so that $s(t) = A\sin(2\pi F_0 t + \phi)$. We model ϕ as a random variable uniformly distributed over $[-a, a]$ or $\phi \sim \mathcal{U}[-a, a]$. In addition, due to observation noise, we observe the samples

$$x(t_n) = A\sin(2\pi F_0 t_n + \phi) + w[n]$$

where $w[n]$ is modeled as white Gaussian noise with variance σ^2 and is independent of ϕ. The mean squared error (MSE) of $\hat{F}_0$ as given by (1) is now found. We need not consider (2) since $x(t_n) \neq 0$ with probability one. The mean squared error of $\hat{F}_0 = F_{\text{MAX}} \Sigma_{n=1}^{N} \hat{b}_n 2^{-n}$ is

$$\begin{aligned} \text{MSE}(\hat{F}_0) &= E[(\hat{F}_0 - F_0)^2] \\ &= E_\phi E_{w|\phi}[(\hat{F}_0 - F_0)^2] \\ &= E_\phi E_w[(\hat{F}_0 - F_0)^2] \end{aligned}$$

where E_ϕ denotes the expected value with respect to the PDF of ϕ, and E_w denotes the expectation with respect to the $w[n]$ samples. However

$$E_w[(\hat{F}_0 - F_0)^2] = \text{var}_w(\hat{F}_0) + (E_w(\hat{F}_0) - F_0)^2.$$

Since $\hat{b}_n$ will be 1 when $x(t_n) < 0$ and 0 otherwise, we have that $\Pr[\hat{b}_n = 1] = \Pr[x(t_n) < 0] = Q((A/\sigma)\sin(2\pi F_0 t_n + \phi)) \triangleq p_n(\phi)$ and therefore, conditioned on $\phi, \hat{b}_n$ is Bernoulli with probability $p_n(\phi)$. $Q(u)$ is defined as $Q(u) = \int_u^\infty (1/\sqrt{2\pi})e^{-(1/2)t^2}\,dt$. Hence

$$\begin{aligned} \text{var}_w(\hat{F}_0) &= F_{\text{MAX}}^2 \sum_{n=1}^{N} \text{var}_w(\hat{b}_n) 2^{-2n} \\ &= F_{\text{MAX}}^2 \sum_{n=1}^{N} p_n(\phi)(1 - p_n(\phi)) 2^{-2n} \end{aligned}$$

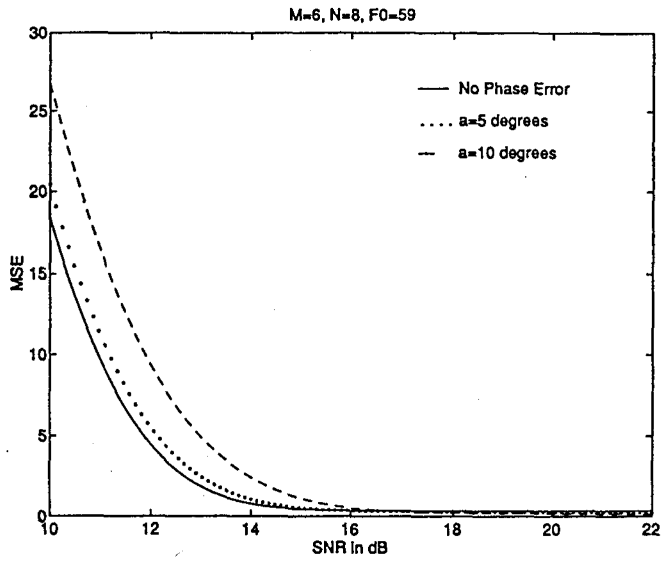

Fig. 1. MSE for a frequency of $F_0 = 59\ Hz$.

and

$$E_w(\hat{F}_0) = F_{\text{MAX}} \sum_{n=1}^{N} p_n(\phi) 2^{-n}.$$

Since, $\phi \sim \mathcal{U}[-a, a]$, we finally have that

$$\begin{aligned} \text{MSE}(\hat{F}_0) = &\frac{1}{2a}\int_{-a}^{a} F_{\text{MAX}}^2 \sum_{n=1}^{N} p_n(\phi)(1 - p_n(\phi)) 2^{-2n}\, d\phi \\ &+ \frac{1}{2a}\int_{-a}^{a} \left(F_{\text{MAX}} \sum_{n=1}^{N} p_n(\phi) 2^{-n} - F_0\right)^2 d\phi \end{aligned} \tag{7}$$

where

$$p_n(\phi) = Q\left(\frac{A}{\sigma}\sin(2\pi F_0 t_n + \phi)\right). \tag{8}$$

As an example, the MSE is shown in Fig. 1 for a frequency of $F_0 = 59$ Hz. The maximum frequency was chosen to be $F_{\text{MAX}} = 64$ Hz ($M = 6$) and $N = 8$ bits of resolution were selected.

B. Observation Interval

Since the sample times are exponential, the length of time over which we must observe $x(t)$ can become large. The observation interval for N bits is $T = 2^{N-M-1} = 2^{N-1}/F_{\text{MAX}}$ A given relative bias error due to bit truncation or $|E(\hat{F}_0) - F_0|/F_0 = \epsilon$ may be obtained if we choose N bits where

$$\frac{2^{-N} F_{\text{MAX}}}{F_0} = \epsilon.$$

(This ignores the bias due to noise.) Since $2T = 2^N/F_{\text{MAX}}$, we have

$$T = \frac{1}{2\epsilon F_0}. \tag{9}$$

The observation time is determined by the accuracy desired and the minimum frequency to be estimated.

IV. Discussion

The approach described is easily implemented in digital hardware using a clock running at a rate of F_{MAX} cycles/s and a binary counter to obtain the exponential sample times. The samples are taken when the counter contains 1, 2, 4, 8, $\cdots$. To ensure a sinusoidal phase of zero, the counter is initialized when $x(t)$ crosses zero in an upwards direction. From (2), the probability of $x(t_n) = 0$ and, therefore, of terminating the expansion is zero in the presence of noise. Thus, if f_0 is a dyadic rational, whose binary expansion should terminate, noise effects may cause the expansion to continue. To avoid this it may be better to truncate the expansion at $n = n_0$ if $|x(t_{n_0})| < \delta$, where δ is a small number.

Last, the estimation method may be used to measure the frequency of other periodic waveforms such as square waves, for example. The only requirement is that we should be able to determine from the waveform if θ_n takes on values as per (5). In addition, the speed of a motor or rate of revolution of any constant rotation object may be ascertained as described.

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Methods for Chaotic Signal Estimation

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Abstract

A dynamic programming algorithm and a suboptimal but computationally efficient method for estimation of a chaotic signal in white Gaussian noise are proposed. The nonlinear map is assumed known so that only the initial condition need be estimated. Computer simulations confirm that both approaches produces efficient estimates at high signal-to-noise ratios.

1 Problem Statement

The problem is to estimate a chaotic signal $s[n]$ that is embedded in white Gaussian noise $w[n]$ of variance σ^2. Hence, the data model is

$$x[n] = s[n] + w[n] \quad n = 0, 1, \ldots, N-1, \tag{1}$$

where $s[n]$ is a deterministic signal generated according to the nonlinear and non-invertible map

$$s[n] = f(\, s[n-1]\,)\,. \tag{2}$$

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The map $f : [0, 1] \to [0, 1]$ is assumed known and to be unimodal. As a result, it has two preimages. Typical examples of such maps are the logistic and tent maps [1]. We furthermore assume that the initial condition $s[0]$ is unknown so that the problem of estimation of $s[n]$ reduces to that of estimation of only the initial condition. Once this estimate is available the entire signal may be estimated, at least in theory, using (2). In practice, because of the extreme sensitivity to errors, (2) cannot be used to propagate signals in the forward direction. Instead, we rely on the parameterization described in [2] whereby the sequence $\{s[0], s[1], \ldots, s[N-1]\}$ is replaced by $\{p_0, p_1, \ldots, p_{N-2}, s[N-1]\}$. The sequence $\{p_n\}_{n=0}^{N-2}$ is the itinerary, or

$$p_n = \begin{cases} 0 & \text{if } \; 0 \leq s[n] < c \\ 1 & \text{if } \; c \leq s[n] < 1 \end{cases},$$

where c is the value of x for which $f(x)$ is maximum. Hence, the itinerary can be used to determine the appropriate preimages of a chaotic signal when propagated backward. As an example, for the logistic map or $f(x) = 4\,x\,(1-x)$, the function is

$$f_p^{-1}(x) = \frac{1 + (2p-1)\sqrt{1-x}}{2}$$

and consists of the preimages $f_0^{-1}(x) = \frac{1-\sqrt{1-x}}{2}$ and $f_1^{-1}(x) = \frac{1+\sqrt{1-x}}{2}$. Thus, we can write

$$s[n-1] = f_{p_{n-1}}^{-1}(s[n]) \, .$$

2 Halving Method

Before discussing the dynamic programming (DP) approach to the maximum likelihood estimate (MLE) evaluation, we describe a suboptimal but computationally efficient approach. In the process we prove that for a large class of unimodal maps the initial condition can be determined from the itinerary of the map. The argument relies on certain topological conjugacy relations. Now assume that f is

conjugate to the tent map T or

$$T(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases} ,$$

so that, there exists an invertible transformation $h : [0, 1] \to [0, 1]$ such that

$$f = h \circ T \circ h^{-1} . \tag{3}$$

Then, we may write $s[n]$ as

$$\begin{aligned} s[n] &= f^n(s[0]) \\ &= h \circ T^n \circ h^{-1}(s[0]) , \end{aligned}$$

since $f^n = h \circ T^n \circ h^{-1}$ and f^n is the n fold composition of f. Let $s'[0] = h^{-1}(s[0])$ so that,

$$T^n(s'[0]) = h^{-1}(s[n]) .$$

Now use the substitution property or $T^n = T \circ S^{n-1}$ [3], where S is the Bernoulli shift

$$S(x) = 2\,x \bmod 1 \quad 0 \leq x \leq 1 ,$$

to yield

$$T \circ S^{n-1}(s'[0]) = h^{-1}(s[n]) .$$

If $s'[0]$ is represented in the binary format $s'[0] = \sum_{k=1}^{\infty} b_k 2^{-k} = 0.b_1 b_2 \ldots$, (where we use a terminating expansion for the dyadic rationals), we have that

$$S^{n-1}(s'[0]) = 0.b_n b_{n+1} \cdots .$$

Hence, it follows that

$$T(\, 0.b_n b_{n+1} \ldots) = h^{-1}(s[n]) .$$

The effect of the tent map is to shift left by one bit if the argument is $0 \leq x < 1/2$ and shift left and complement (due to $1 - x$ factor) if $1/2 \leq x \leq 1$. As a result,

$$\begin{aligned} h^{-1}(s[n]) &= 0.b_{n+1} b_{n+2} \ldots \quad \text{if } b_n = 0, \\ &= 0.\bar{b}_{n+1} \bar{b}_{n+2} \ldots \quad \text{if } b_n = 1, \end{aligned}$$

where the overbar denotes the complement bit. This says that

$$b_{n+1} = \begin{cases} 0 & \text{if} \quad 0 \le h^{-1}(s[n]) < \frac{1}{2} & \text{and} \quad b_n = 0 \\ 1 & \text{if} \quad \frac{1}{2} \le h^{-1}(s[n]) < 1 & \text{and} \quad b_n = 0 \\ 1 & \text{if} \quad 0 \le h^{-1}(s[n]) < \frac{1}{2} & \text{and} \quad b_n = 1 \\ 0 & \text{if} \quad \frac{1}{2} \le h^{-1}(s[n]) < 1 & \text{and} \quad b_n = 1 \end{cases} .$$

If we now let p_n be the itinerary of $h^{-1}(s[n])$ or $p_n = 0$ if $0 \le h^{-1}(s[n]) < 1/2$ and $p_n = 1$ if $1/2 \le h^{-1}(s[n]) < 1$, we have finally that,

$$b_{n+1} = b_n \oplus p_n \quad ,$$

where $\oplus$ denotes the exclusive OR operation. The recursion begins with $b_1 = p_0$. To summarize, we can determine the initial condition of a map that satisfies the conjugacy relation of (3) as

$$s[0] = h\left(\sum_{n=1}^{\infty} b_n 2^{-n}\right) \quad ,$$

where $b_n = b_{n-1} \oplus p_{n-1}$, $(b_1 = p_0)$ and p_n is the itinerary of $h^{-1}(s[n])$. It is interesting to note that in practice we will have $\{p_0, p_1, \ldots, p_{N-1}\}$ based on the itinerary of the data set $\{s[0], s[1], \ldots, s[N-1]\}$, assuming no noise. Hence, we will have

$$\hat{s}[0] = h\left(\sum_{n=1}^{N} b_n\, 2^{-n}\right) \quad , \tag{4}$$

so that the estimate of $h^{-1}(s[0])$ will be in error by at most $1/2^N$. We will call the estimation of $s[0]$ by (4) the *halving method* since as each itinerary value is obtained, the interval in which $s'[0]$ must reside is halved.

As an example, for the chaotic logistic map $f(x) = 4x(1-x)$ it is known that $h(x) = \sin^2(\frac{\pi}{2}x)$ and $h^{-1}(x) = \frac{1}{\pi}\arccos(1-2x)$. Since h^{-1} maps $[0, 1/2)$ onto $[0, 1/2)$ and $[1/2, 1)$ onto $[1/2, 1)$ the itineraries of $h^{-1}(s[n])$ and $s[n]$ are the same. Thus, the halving method estimates $s[0]$ as

$$\hat{s}[0] = \sin^2\left(\frac{\pi}{2}\sum_{n=1}^{N} \hat{b}_n\, 2^{-n}\right) \quad , \tag{5}$$

where $\hat{b}_n = \hat{b}_{n-1} \oplus \hat{p}_{n-1}$ and $\hat{p}_n$ is the itinerary of $x[n]$. The performance for this example will be discussed in Section 4. Note that, if we consider $s[n]$ as the

initial iterate, then it depends only on $\{p_n, p_{n+1}, \ldots\}$. Furthermore, $s[n]$ may be determined directly from this itinerary in a similar fashion as per (4). Or since the itinerary $\{p_n, p_{n+1}, \ldots\}$ can be obtained for any $s[n]$ there is a one-to-one correspondence between $s[0]$ and $\{p_0, p_1, \ldots, p_{N-2}, s[N-1]\}$ as already noted in [2].

3 Dynamical Programming

The MLE for the initial condition is obtained as the value of $s[0]$ that minimizes $J = \sum_{n=0}^{N-1} \left(x[n] - s[n]\right)^2$ where $s[n] = f^n(s[0])$. A straightforward minimization requires one to compute $f^n(s[0])$, which leads to computational errors. Rather, we employ a DP approach, which does not require a forward propagation. The method to be described applies to any unimodal map. In particular, for the tent map, which is piece-wise linear, an analytical solution may be found as in [2]. Let

$$J_k\left(s[k]\right) = \sum_{n=0}^{k} \left(x[n] - s[n]\right)^2$$

and also

+

$$s[n] = f^{-1}_{p_n, p_{n-1}, \ldots, p_{k-1}}\left(s[k]\right) \tag{6}$$

be the inverse function composition for $n \leq k-1$. This is defined as

$$\begin{aligned}
s[k-1] &= f^{-1}_{p_{k-1}}(s[k]) \\
s[k-2] &= f^{-1}_{p_{k-2}}(s[k-1]) \\
&= f^{-1}_{p_{k-2}}(f^{-1}_{p_{k-1}}(s[k])) \\
&= f^{-1}_{p_{k-2}, p_{k-1}}(s[k])
\end{aligned}$$

etc.

Now, let

$$I_k(s[k]) = \min_{\{p_0, p_1, \ldots, p_{k-1}\}} J_k(s[k])$$

so that the desired minimization is effected when $k = N-1$ and $I_k(s[N-1])$ is minimized over $s[N-1]$. Thus,

$$I_k(s[k]) = \min_{\{p_0, p_1, \ldots, p_{k-1}\}} \sum_{n=0}^{k} \left(x[n] - s[n]\right)^2$$

$$= \min_{p_{k-1}} \min_{\{p_0,p_1,\ldots,p_{k-2}\}} \sum_{n=0}^{k-1} (x[n]-s[n])^2 + (x[k]-s[k])^2$$
$$= \min_{p_{k-1}} \min_{\{p_0,p_1,\ldots,p_{k-2}\}} \left[J_{k-1}(s[k-1]) + (x[k]-s[k])^2\right]$$
$$= \min_{p_{k-1}} \min_{\{p_0,p_1,\ldots,p_{k-2}\}} \left[J_{k-1}\left(f^{-1}_{p_{k-1}}(s[k])\right) + (x[k]-s[k])^2\right]$$

Since $s[k]$ does not depend on $\{p_0, p_1, \ldots, p_{k-1}\}$ but only on $\{p_k, \ldots, p_{N-2}, s[N-1]\}$, as per (6) we have,

$$I_k(s[k]) = \min_{p_{k-1}} I_{k-1}\left(f^{-1}_{p_{k-1}}(s[k])\right) + (x[k]-s[k])^2 \ , \tag{7}$$

as our DP algorithm. The recursion is computed as a function of $s[k]$ for $k = 1, 2, \ldots, N-1$ with the initialization

$$I_0(s[0]) = (x[0]-s[0])^2 \ .$$

After obtaining $I_{N-1}(s[N-1])$, we find the $s[N-1]$ that minimizes it and denote the minimizing value as $\hat{s}[N-1]$. We then backtrack to find $\{\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_{N-2}\}$. In doing so, the signal sequence is estimated as

$$\hat{s}[n-1] = f^{-1}_{\hat{p}_{n-1}}(\hat{s}[n])$$

for $n = N-1, N-2, \ldots, 1$. Note also that this approach avoids the exponential increase in computational errors since the propagation is along the stable manifold.

4 Computer Simulations Results

It has been shown [4] that the MLE of the initial condition of a one-dimensional chaotic signal is an inconsistent estimator. Hence, the usual asymptotic distribution as $N \to \infty$ does not hold. However, as the SNR becomes large, the asymptotic distribution is valid. In particular, the asymptotic probability density function (PDF) of $\hat{s}[0]$ is

$$\hat{s}[0] \sim \mathcal{N}\left(s[0], I^{-1}(s[0])\right), \tag{8}$$

where $I(s[0])$ is the Fisher information (and hence, $I^{-1}(s[0])$ is the Cramer-Rao lower bound (CRLB) for an unbiased estimator) and is found to be

$$I^{-1}(s[0]) = \frac{\sigma^2}{\sum_{m=0}^{N-1} \beta_m^2} \ , \tag{9}$$

where

$$\beta_m^2 = \begin{cases} 1 & \text{if } m = 0 \\ \prod_{l=0}^{m-1} [f'(s[l])]^2 & \text{if } m > 0 \end{cases}, \tag{10}$$

and f' denotes the derivative of f [5].

We compare the performance of an MLE using a fine grid search (by minimizing $J = \sum_{n=0}^{N-1} (x[n] - s[n])^2$ directly) and that of the two algorithms previously described to the theoretical performance of (8). In particular, the bias and variance are determined using Monte Carlo simulations. Note that the CRLB for a logistic map can be shown to be

$$I^{-1}(s[0]) = \frac{\sigma^2 s[1]}{\sum_{n=0}^{N-1} 4^n s[n+1]}.$$

We used a data record length of $N = 10$ and an initial condition of $s[0] = 0.61$. Some implementation details for the three methods are now discussed. For the grid search MLE we first use a coarse search by dividing up the $[0, 1]$ interval into 1000 points. Then we search over the interval $[\hat{s}_c[0] - 0.001, \hat{s}_c[0] + 0.001]$ using 1000 points where $\hat{s}_c[0]$ is the coarsely obtained point of the minimum. The implementation of the halving method departs slightly from (5). This is because the use of (5) may produce a maximum error of $1/2^N \approx 0.001$, even in the absence of noise. The CRLB, because of its exponential decrease with N, can be much lower than this error. To improve the performance we first obtain the coarse estimate

$$\hat{s}_c[0] = \sin^2\left(\frac{\pi}{2}\left(\sum_{n=1}^{N} \hat{b}_n 2^{-n} + 2^{-(N+1)}\right)\right)$$

where the addition of the term $2^{-(N+1)}$ has the effect of choosing the midpoint of the interval $\left(\sum_{n=1}^{N} \hat{b}_n 2^{-n}, \sum_{n=1}^{N} \hat{b}_n 2^{-n} + \frac{1}{2^N}\right)$. Then, a fine search (100 points) over the interval $(\hat{s}_c[0] - 2^{-N}, \hat{s}_c[0] + 2^{-N})$ is made by minimizing J, as in the MLE implementation. Note that the length is twice that of the coarsely obtained interval. The fine search is repeated twice over successively smaller intervals of 100 points. The DP implementation of the MLE uses 500 points for the state $s[k]$. For the search over $s[N-1]$ we use an initial quantization of 500 points to locate a coarse estimate of $\hat{s}[N-1]$. We then choose a finer grid for $\hat{s}[N-1]$ centered about

the previous estimate as well as the same estimates $\{\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_{N-2}\}$ obtained previously.

The results are shown in Figure 1. Above the threshold of about 40dB all three methods produce unbiased estimates that achieve the CRLB. (The bias, although not shown, was negligible above the threshold). This is in accordance with the theoretical results of [4]. Of the three methods the halving approach is computationally the simplest. If enough data points are available, the fine search employed for the halving method may be eliminated. This is because the error will be at most $1/2^N$ and so the estimate, which although not attaining the CRLB, will be accurate enough for most practical purposes.

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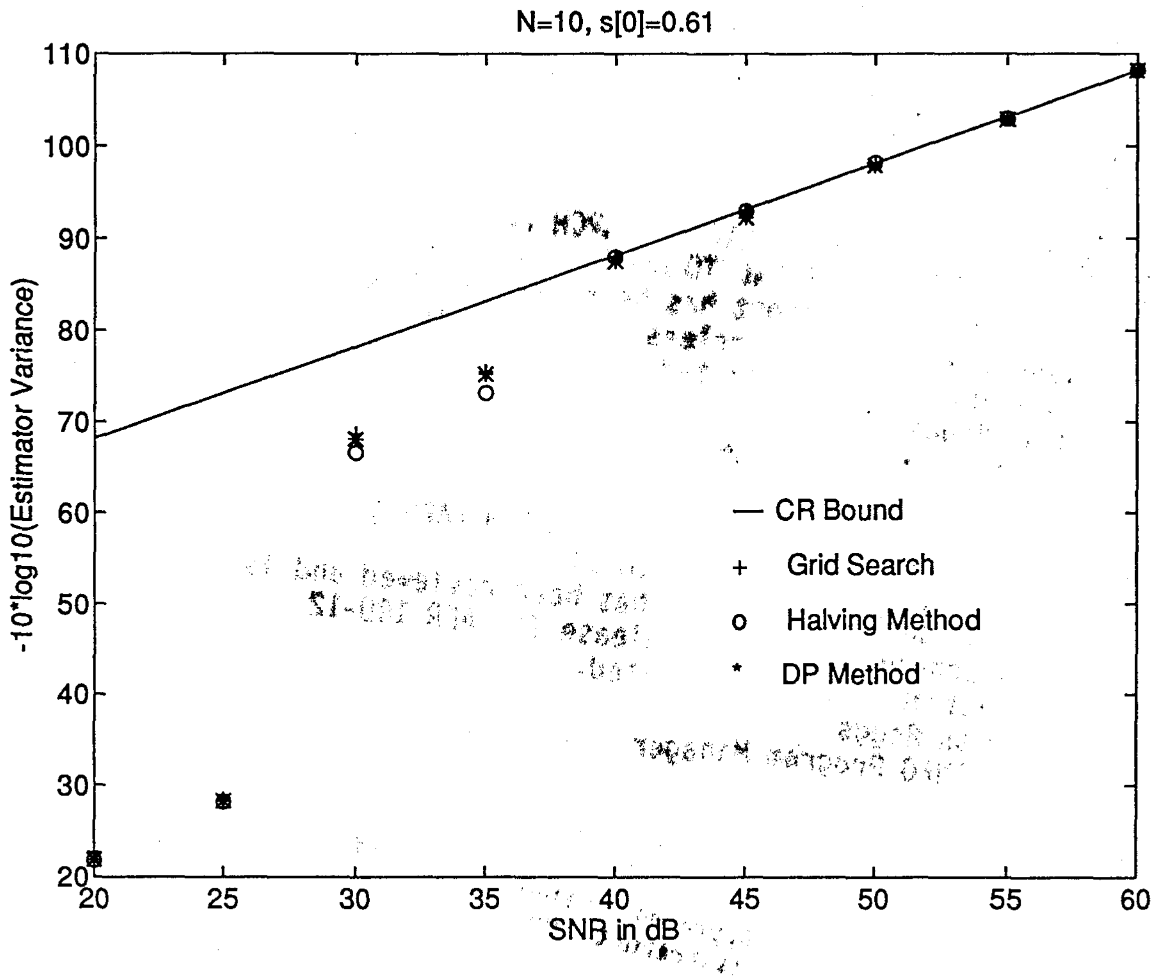

Figure 1: Performance of Different Methods